

# BOUNDARY BEHAVIOR OF INVARIANT GREEN'S POTENTIALS ON THE UNIT BALL IN $\mathbb{C}^n$

K. T. HAHN AND DAVID SINGMAN

**ABSTRACT.** Let  $p(z) = \int_B G(z, w) d\mu(w)$  be an invariant Green's potential on the unit ball  $B$  in  $\mathbb{C}^n$  ( $n \geq 1$ ), where  $G$  is the invariant Green's function and  $\mu$  is a positive measure with  $\int_B (1 - |w|^2)^n d\mu(w) < \infty$ .

In this paper, a necessary and sufficient condition on a subset  $E$  of  $B$  such that for every invariant Green's potential  $p$ ,

$$\liminf_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n p(z) = 0, \quad e = (1, 0, \dots, 0) \in \partial B, \quad z \in E,$$

is given. The condition is that the capacity of the sets  $E \cap \{z \in B \mid |z - e| < \varepsilon\}$ ,  $\varepsilon > 0$ , is bounded away from 0. The result obtained here generalizes Luecking's result, see [L], on the unit disc in  $\mathbb{C}$ .

**1. Introduction.** Let  $E$  be a subset of  $B$ , the unit ball in  $\mathbb{C}^n$ ,  $n \geq 1$ . An invariant Green's potential is a function on  $B$  of the form

$$p(z) = \int_B G(z, w) d\mu(w),$$

where  $G$  is the invariant Green's function (see 2.11b) and  $\mu$  is a positive measure such that  $\int_B (1 - |w|^2)^n d\mu(w) < \infty$ . In this paper we give a necessary and sufficient condition on  $E$  such that, for every invariant Green's potential  $p$ ,

$$\liminf_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n p(z) = 0,$$

where  $e = (1, 0, \dots, 0)$ . It is that the capacity of the sets  $E \cap \{z \in B \mid |z - e| < \varepsilon\}$ ,  $\varepsilon > 0$ , be bounded away from 0. Here, capacity refers to the capacity with respect to the potential theory based on the Laplace-Beltrami operator on the ball with respect to the Bergman metric. See (4.1). This solves a problem posed in [HS], where the result was proved in the special case of  $E = \{(z, z') \in \mathbb{C} \times \mathbb{C}^{n-1} \mid \operatorname{Im} z = 0, (\operatorname{Re} z)^2 + |z'|^2 < 1\}$ . Our result generalizes a result of Luecking [L] on the unit disc in  $\mathbb{C}$ .

**2. Preliminaries.** For  $z, w \in \mathbb{C}^n$  let  $\langle z, w \rangle = \sum_{\alpha=1}^n z_\alpha \bar{w}_\alpha$ ,  $|z|^2 = \langle z, z \rangle$ . For  $0 < r \leq 1$ , let  $B_r = \{z \in B \mid |z| < r\}$ ,  $S_r = \{z \in B \mid |z| = r\}$ ,  $B = B_1$ , and  $S = S_1$ . Let  $\sigma$  be the rotation-invariant positive Borel measure on  $S$  with  $\sigma(S) = 1$ . Put  $e = (1, 0, \dots, 0)$ . For each  $a, z \in B$  let

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1 - |a|^2} Q_a(z)}{1 - \langle z, a \rangle},$$

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where  $P_a(z) = \langle z, a \rangle a / |a|^2$  and  $Q_a(z) = z - P_a(z)$ . Then one has

$$(2.1) \quad 1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2} \quad [\mathbf{R}, \text{Theorem 2.2.2}].$$

The Bergman metric on the ball  $B$  is given by

$$(2.2a) \quad ds_B^2(z) = \sum_{\alpha, \beta=1}^n g_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta,$$

where

$$(2.2b) \quad g_{\alpha\bar{\beta}} = \frac{n+1}{(1 - |z|^2)^2} \{ (1 - |z|^2) \delta_{\alpha\beta} + \bar{z}_\alpha z_\beta \}$$

[St, p. 23]. The corresponding invariant volume element [K, p. 17] is thus

$$(2.2c) \quad d\lambda(z) = \frac{n+1}{(1 - |z|^2)^{n+1}} dm(z),$$

where  $dm$  denotes the restriction of Lebesgue measure to  $B$ . For each  $f \in L^1(\lambda)$ ,  $a \in B$ ,  $\lambda$  satisfies

$$(2.3) \quad \int_B f \circ \varphi_a d\lambda = \int_B f d\lambda$$

[R, Theorem 2.2.6]. The inverse of  $(g_{\alpha\bar{\beta}})$  is  $(g^{\alpha\bar{\beta}})$ , where

$$g^{\alpha\bar{\beta}} = \frac{1 - |z|^2}{n+1} (\delta_{\alpha\beta} - \bar{z}_\alpha z_\beta).$$

The Laplace-Beltrami operator of the metric is

$$\begin{aligned} \Delta &= 4 \sum_{\alpha, \beta=1}^n g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial \bar{z}_\alpha \partial z_\beta} \\ &= \frac{4}{n+1} (1 - |z|^2) \sum_{\alpha, \beta=1}^n (\delta_{\alpha\beta} - \bar{z}_\alpha z_\beta) \frac{\partial^2}{\partial \bar{z}_\alpha \partial z_\beta} \end{aligned}$$

[St, p. 27].

A  $C^2$  function defined on an open subset of  $B$  that is annihilated by  $\Delta$  will be called *harmonic*. The set of functions harmonic on open subsets of  $B$  forms a BreLOT harmonic space [H, Théorème 34.1]. We will make some use of the definitions and results available in such a general setting. For details see [B].

Let  $\Omega \subset B$  be open. A function  $u$  is said to be *superharmonic* on  $\Omega$  if (i)  $u: \Omega \rightarrow (-\infty, \infty]$ , (ii)  $u$  is lower semicontinuous, (iii) for each  $a \in \Omega$ , there exists  $r(a) > 0$  such that for all  $0 < r \leq r(a)$ ,

$$(2.4) \quad u(a) \geq \int_S u(\varphi_a(r\zeta)) d\sigma(\zeta),$$

and (iv) none of the integrals in (iii) is  $\infty$  [U, Definition 1.15].

It was observed in [UT, Proposition 1.6] that if  $u$  is superharmonic on  $\Omega$  then (2.4) holds for all  $r > 0$  such that  $\varphi_a(\bar{B}_r) \subset \Omega$ .

We wish to see that the above definition agrees with the definition of superharmonic in a BreLOT space [H, p. 427, Definition A]. If  $u$  is  $C^2$ , then both definitions

of superharmonic are equivalent to  $\Delta u \leq 0$  [H, Proposition 34.1; U, p. 505 or UT, Proposition 1.18]. Since it is clear that the limit of an increasing sequence of BreLOT superharmonic functions is either identically  $\infty$  or BreLOT superharmonic, the result will follow from

**PROPOSITION 2.1.** *Let  $u$  be superharmonic on  $\Omega$ . Let  $\omega$  be an open, relatively compact subset of  $\Omega$ . Then there is an increasing sequence of  $C^\infty(\omega)$  functions, superharmonic on  $\omega$ , with limit  $u$ .*

*Note.* This was proved for  $\Omega = B$  in [UT, Theorem 1.25]. We include the proof for the reader's convenience.

**PROOF.** We shall see that the proof depends only on the values of  $u$  on some compact neighborhood of  $\omega$ . Thus, since constants are harmonic, we may assume for the remainder of the proof that  $u$  is nonnegative.

For  $f, g \geq 0$ , define, for  $a \in B$ ,

$$f * g(a) = \int_B f(z)g(\varphi_a(z)) d\lambda(z)$$

[U, Definition 2.1].

Let  $\omega_1$  be a relatively compact subset of  $\Omega$  with  $\bar{\omega} \subset \omega_1$ . There exists  $r_1 > 0$  such that

$$(2.5) \quad \varphi_a(\bar{B}_{r_1}) \subset \omega_1 \quad (\text{all } a \in \omega).$$

Since  $\varphi_a^{-1} = \varphi_a$ , this says  $z$  is in  $\omega_1$  whenever  $|\varphi_a(z)| \leq r_1$ .

Let  $\chi \geq 0$  be radial and  $C^\infty$  with support in  $B_{r_1}$  and suppose  $\int_B \chi d\lambda = 1$ . Since

$$(2.6) \quad u * \chi(a) = \int_B u(z)\chi(\varphi_a(z)) d\lambda(z) = \int_{B_{r_1}} u(\varphi_a(z))\chi(z) d\lambda(z),$$

the value of  $u * \chi$  on  $\omega$  depends only on the value of  $u$  on  $\omega_1$ . The first equality in (2.6) shows  $u * \chi \in C^\infty(\omega)$ . Integrating the second in polar coordinates shows  $u * \chi \leq u$  on  $\omega$ .

Fix  $a \in \omega$ . Let  $\omega_2$  be a relatively compact open subset of  $\Omega$  such that  $\bar{\omega}_1 \subset \omega_2$ . Choose  $0 < r_2 < r_1$  such that  $\varphi_b(\bar{B}_{2r_2}) \subset \omega_2$  (all  $b \in \omega_1$ ). Let  $\varepsilon > 0$ . Since  $(u * \chi) \circ \varphi_a$  is uniformly continuous on  $\bar{B}_{r_2}$ , there exists  $0 < \delta < r_2$  such that

$$(2.7) \quad |(u * \chi)(\varphi_a(r\zeta)) - (u * \chi)(\varphi_a(s\zeta))| < \varepsilon$$

for all  $0 < r < r_2$ ,  $\zeta \in S$ , and  $|s - r| < \delta$ . Fix any  $r > 0$  with  $r < r_2$ . Let  $h \geq 0$  be  $C^\infty$ , radial,  $\int_B h d\lambda = 1$  with support in  $\{z | r - \delta < |z| < r + \delta\}$ . Then, with a similar proof as above,

$$(2.8) \quad u * h(\zeta) \leq u(\zeta) \quad (\text{all } \zeta \text{ in } \omega_1).$$

Thus, by [U, Proposition 2.2],

$$\begin{aligned} (u * \chi) * h(a) &= u * (\chi * h)(a) = u * (h * \chi)(a) = (u * h) * \chi(a) \\ &= \int_{B_{r_1}} (u * h)(\varphi_a(z))\chi(z) d\lambda(z) \\ (2.9) \quad &\leq \int_{B_{r_1}} u(\varphi_a(z))\chi(z) d\lambda(z) \\ &= u * \chi(a). \end{aligned}$$

The inequality in (2.9) follows from (2.5) and (2.8). From (2.7),

$$\begin{aligned} & |(u * \chi) * h(a) - \int_S (u * \chi)(\varphi_a(r\zeta)) d\sigma(\zeta)| \\ & \leq \int_{|s|=r-\delta}^{r+\delta} \int_S |(u * \chi)(\varphi_a(s\zeta)) - (u * \chi)(\varphi_a(r\zeta))| \frac{s^{2n-1}}{(1-s^2)^{n+1}} h(s) d\sigma(\zeta) ds \\ & < \varepsilon. \end{aligned}$$

Thus

$$(2.10) \quad \int_S (u * \chi)(\varphi_a(r\zeta)) d\sigma(\zeta) < \varepsilon + (u * \chi) * h(a) \leq \varepsilon + (u * \chi)(a)$$

by (2.9). Since  $\varepsilon$  is arbitrary, (2.10) implies  $u * \chi$  is superharmonic on  $\omega$ .

Let  $\{r_j\}$  be a sequence of real numbers decreasing to 0. Choose, for each  $j$ ,  $\chi_j \geq 0$ , radial,  $C^\infty$ ,  $\int_B \chi_j = 1$  with support in  $\{z | r_{j+1} < |z| < r_j\}$ . Then we have seen  $\{u * \chi_j\}$  is a sequence of functions superharmonic on  $\omega$ . The fact that they increase to  $u$  on  $\omega$  was shown in [U, Lemma 2.13]. This completes the proof.

Let  $S^+$  denote the space of nonnegative superharmonic functions on  $B$ . An element of  $S^+$  which majorizes no positive harmonic functions is called a *potential* [H, p. 427]. These are precisely of the form

$$(2.11a) \quad G\mu(z) = \int_B G(z, w) d\mu(w),$$

where

$$(2.11b) \quad G(z, w) = \frac{n+1}{2n} \int_{|\varphi_z(w)|}^1 \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt$$

is the Green's function on  $B$  and  $\mu$  is a positive measure for which  $G\mu(z) \not\equiv \infty$  [U, Theorem 2.16]. It should be pointed out that although the definition of Green's function of  $B$  is given in [U], the actual computation of the Green's function of  $B$  was carried out in [HM] as a special case of more general classical Cartan domains.

Let  $0 < c < 1$  be fixed. It is shown in [HS] that there are constants  $c_1$  and  $c_2 = c_2(c)$  such that

$$(2.12a) \quad G(z, w) \geq c_1(1 - |\varphi_z(w)|^2)^n \quad (\text{all } z, w \text{ in } B),$$

$$(2.12b) \quad G(z, w) \leq c_2(1 - |\varphi_z(w)|^2)^n \quad (\text{if } |\varphi_z(w)| \geq c).$$

It follows from this and (2.1) that  $G\mu$  is a potential if and only if

$$(2.13) \quad \int_B (1 - |w|^2)^n d\mu(w) < \infty.$$

For  $E \subset B$  and any  $v \in S^+$ , the *reduced function* and its *regularization* are defined by

$$R_v^E(z) = \inf\{w(z) | w \in S^+, w \geq v \text{ on } E\}$$

and

$$\hat{R}_v^E(z) = \liminf_{\zeta \rightarrow z} R_v^E(\zeta),$$

respectively [H, p. 433 or B, Definition 9]. Then  $\hat{R}_v^E$  is harmonic on  $B \setminus \bar{E}$  and superharmonic on  $B$ . Moreover, we have the following relations:

$$(2.14a) \quad \hat{R}_v^E \leq R_v^E \leq v \quad \text{on } B,$$

$$(2.14b) \quad R_v^E = v \quad \text{on } E,$$

and

$$(2.14c) \quad \hat{R}_v^E = R_v^E \quad \text{on } B \setminus \bar{E} \text{ and the interior of } E.$$

Thus,  $\hat{R}_v^E = R_v^E$  in case  $E$  is open. If  $E$  is relatively compact in  $B$ , then  $\hat{R}_v^E$  is a potential.

A set  $E \subset B$  is *polar* if there is a potential which is  $\infty$  on  $E$  [H, p. 434 or B, Definition 21]. The polar sets defined here coincide with the corresponding notion in the classical potential theory [H, Théorème 36.1].

The most important result concerning polar sets is the Cartan-Brelot convergence theorem which states that if  $\{v_k\}$  is a decreasing sequence in  $S^+$ , then

$$\left( \lim_{k \rightarrow \infty} v_k(z) \right)^\wedge = \liminf_{\zeta \rightarrow z} \left( \lim_{k \rightarrow \infty} v_k(\zeta) \right)$$

is superharmonic and equals  $\lim_{k \rightarrow \infty} v_k(z)$  except at most on a polar set [H, p. 436, Theorem 27]. The Topological Lemma of Choquet [D, Lemma A, VIII.3] implies

$$(2.15) \quad \hat{R}_v^E(z) = R_v^E(z),$$

except perhaps on a polar set. The proof of the convergence theorem makes use of the *Domination Principle* [H, p. 436, Axiome D, and Corollaire to Théorème 36.2] which states that if  $v \in S^+$ ,  $G\mu$  is a finite potential satisfying  $v \geq G\mu$  on the support of  $\mu$ , then  $v \geq G\mu$  holds on  $B$ . As a consequence, a set  $E \subset B$  is polar if and only if  $\hat{R}_1^E \equiv 0$  or equivalently  $R_1^E(z) = 0$  for one  $z \in B$ .

For any  $E \subset B$ ,

$$(2.16a) \quad R_1^E(z) = \inf \{ R_1^U(z) \mid U \text{ open, } U \supset E \}$$

[H, p. 434]. Hence, by the Topological Lemma of Choquet, there is a decreasing sequence of open sets  $\{U_k\}$ ,  $U_k \supset E$ , such that, for all  $z \in B$ ,

$$(2.16b) \quad \hat{R}_1^E(z) = \left( \lim_{k \rightarrow \infty} R_1^{U_k}(z) \right)^\wedge.$$

For a Borel set  $E \subset B$ ,

$$(2.17a) \quad \hat{R}_1^E(z) = \sup \{ \hat{R}_1^K(z) \mid K \text{ compact, } K \subset E \}$$

[H, p. 434, Théorème 8]. Hence, by [D, Theorem A, VIII.2], there is an increasing sequence of compact sets  $\{K_j\}$  contained in  $E$  such that, for all  $z \in B$ ,

$$(2.17b) \quad \hat{R}_1^E(z) = \lim_{j \rightarrow \infty} \hat{R}_1^{K_j}(z).$$

**3. Energy.** Let  $\mu$  and  $\nu$  be two positive measures on  $B$  such that  $G\mu$  and  $G\nu$  are the corresponding potentials. Define the *mutual energy* of  $\mu$  and  $\nu$  by

$$(3.1) \quad [\mu, \nu] = \int_B G\mu(z) d\nu(z).$$

The *energy* of  $\mu$  (or  $G\mu$ ) is  $\|\mu\| = [\mu, \mu]^{1/2}$ . Fubini's theorem and the symmetry of  $G$  (deduced from (2.1)) implies

$$(3.2) \quad [\mu, \nu] = [\nu, \mu].$$

LEMMA 3.1. *Let  $\{G\mu_j\}$ ,  $\{G\nu_j\}$  be increasing sequences of potentials with limits  $G\mu$  and  $G\nu$ , respectively. Then  $\{[\mu_j, \nu_j]\}$  is an increasing sequence with limit  $[\mu, \nu]$ .*

PROOF. Since

$$\begin{aligned} \int_B G\mu_j d\nu_j &\leq \int_B G\mu_{j+1} d\nu_j = \int_B G\nu_j d\mu_{j+1} \\ &\leq \int_B G\nu_{j+1} d\mu_{j+1} = \int_B G\mu_{j+1} d\nu_{j+1}, \end{aligned}$$

the sequence  $\{[\mu_j, \nu_j]\}$  is indeed increasing. For each  $k > 0$ ,

$$\begin{aligned} \lim_{j \rightarrow \infty} [\mu_j, \nu_j] &\geq \liminf_{j \rightarrow \infty} \int_B G\mu_k d\nu_j = \liminf_{j \rightarrow \infty} \int_B G\nu_j d\mu_k \\ &= \int_B G\nu d\mu_k = \int_B G\mu_k d\nu. \end{aligned}$$

Letting  $k \rightarrow \infty$  gives  $\lim_{j \rightarrow \infty} [\mu_j, \nu_j] \geq [\mu, \nu]$ . The opposite inequality is obvious.

An improvement of Lemma 3.1 will be given later in Corollary 3.4.

PROPOSITION 3.2. *Let  $G\mu$  and  $G\nu$  be two potentials. Then*

$$(3.3) \quad [\mu, \nu] \leq \|\mu\| \|\nu\|.$$

PROOF. Let  $u = G\mu$  and  $v = G\nu$ . Suppose first that  $u$  and  $v$  are  $C^\infty$  and  $\mu$  and  $\nu$  are supported by  $B_{r_0}$ ,  $0 < r_0 < 1$ . Let  $\{\psi_j\}$  and  $\{\varphi_j\}$  be sequences in  $C_c^\infty(B)$ ,  $\{\psi_j\}$  increasing to 1 on  $B$  and  $\varphi_j \equiv 1$  on a neighborhood of the support of  $\psi_j$ . Then

$$\begin{aligned} - \int_B \psi_j u d\nu &= \int_B v \Delta(\psi_j u) d\lambda = \int_B \varphi_j v \Delta(\psi_j u) d\lambda \\ &= \int_B \psi_j u \Delta(\varphi_j v) d\lambda = \int_B \psi_j u \Delta v d\lambda. \end{aligned}$$

Here the first equality follows from Theorem 2.5 of [U] and the third from Proposition 2.4 of [U]. Letting  $j \rightarrow \infty$ , we obtain

$$(3.4) \quad [\mu, \nu] = - \int_B u \Delta v d\lambda.$$

Let  $r_0 < r < 1$ . By Green's identity [S, 92.5],

$$\int_{S_r} u \frac{\partial v}{\partial n} d\tau - \int_{B_r} u \Delta v d\lambda = \int_{B_r} (\text{grad } u, \text{grad } v) d\lambda,$$

where  $d\tau$  is the volume element determined by the metric (2.2) on  $\partial B_r = S_r$ , and

$$(\text{grad } u, \text{grad } v) = 4 \operatorname{Re} \sum_{\alpha, \beta} g^{\alpha\bar{\beta}} \frac{\partial u}{\partial \bar{z}_\alpha} \frac{\partial v}{\partial z_\beta}.$$

We claim

$$(3.5) \quad \lim_{r \rightarrow 1} \int_{\partial B_r} u \frac{\partial v}{\partial n} d\tau = 0.$$

For  $|z| = r$ ,

$$\begin{aligned} u(z) &= \int_{B_{r_0}} G(z, w) d\mu(w) \leq c_2 \int_{B_{r_0}} (1 - |\varphi_z(w)|^2)^n d\mu(w) \\ (3.6) \quad &= c_2 \int_{B_{r_0}} \frac{(1 - |z|^2)^n (1 - |w|^2)^n}{|1 - \langle z, w \rangle|^{2n}} d\mu(w) = O(1 - r^2)^n. \end{aligned}$$

The inequality in (3.6) follows from (2.12b), while the last equality is a consequence of  $u$  being a potential.

The derivative of  $v$  in the outward normal direction along  $S_r$  is given by

$$\begin{aligned} (3.7) \quad \frac{\partial v}{\partial n} &= 2 \operatorname{Re} \left( \sum_{\alpha, \beta} g^{\alpha\bar{\beta}} \frac{\partial v}{\partial z_\beta} \frac{\partial f}{\partial \bar{z}_\alpha} \right) / \left( \sum_{\alpha, \beta} g^{\alpha\bar{\beta}} \frac{\partial f}{\partial z_\beta} \frac{\partial f}{\partial \bar{z}_\alpha} \right)^{1/2} \\ &= \frac{(\operatorname{grad} f, \operatorname{grad} v)}{\|\operatorname{grad} f\|}, \end{aligned}$$

where  $f(z) = \sum_{\alpha} z_{\alpha} \bar{z}_{\alpha} - r^2$ .

After some calculations, we obtain

$$(3.8a) \quad \|\operatorname{grad} f\|^2 = \frac{4}{n+1} r^2 (1 - r^2)^2,$$

$$(3.8b) \quad (\operatorname{grad} f, \operatorname{grad} v) = \frac{4}{n+1} (1 - |z|^2)^2 \operatorname{Re} \sum_{\alpha=1}^n z_{\alpha} \frac{\partial v}{\partial z_{\alpha}}.$$

So, from (3.7) and (3.8),

$$(3.9) \quad \frac{\partial v}{\partial n} = \frac{2}{\sqrt{n+1}} \frac{1 - r^2}{r} \operatorname{Re} \left( \sum_{\alpha=1}^n z_{\alpha} \frac{\partial v}{\partial z_{\alpha}} \right).$$

But a straightforward computation shows

$$\operatorname{Re} \left( \sum_{\alpha=1}^n z_{\alpha} \frac{\partial v}{\partial z_{\alpha}} \right) = O(1 - r^2)^{n-1}.$$

Thus,

$$(3.10) \quad \partial v / \partial n = O(1 - r^2)^n.$$

For  $z \in S_r$ ,  $d\lambda(z)$  is reduced to

$$(3.11) \quad d\tau(z) = \frac{n+1}{(1 - r^2)^{n+1}} r^{2n-1} d\sigma(z).$$

So, from (3.6), (3.11) and (3.12),

$$\int_{S_r} u \frac{\partial v}{\partial n} d\tau = O(1 - r^2)^{n-1} \rightarrow 0 \quad \text{as } r \rightarrow 1 \text{ if } n \geq 2.$$

This proves the claim in (3.5).

Now

$$\begin{aligned}
 |(\text{grad } u, \text{grad } v)| &= \left| 4 \operatorname{Re} \sum_{\alpha, \beta} g^{\alpha \bar{\beta}} \frac{\partial v}{\partial z_{\beta}} \frac{\partial u}{\partial \bar{z}_{\alpha}} \right| \\
 &\leq \left( \sum_{\alpha, \beta} 4g^{\alpha \bar{\beta}} \frac{\partial u}{\partial z_{\beta}} \frac{\partial u}{\partial \bar{z}_{\alpha}} \right)^{1/2} \left( \sum_{\alpha, \beta} 4g^{\alpha \bar{\beta}} \frac{\partial v}{\partial z_{\beta}} \frac{\partial v}{\partial \bar{z}_{\alpha}} \right)^{1/2} \\
 &= \|\text{grad } u\| \|\text{grad } v\|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_{S_r} u \frac{\partial v}{\partial n} d\tau - \int_{B_r} u \Delta v d\lambda &\leq \int_{B_r} \|\text{grad } u\| \|\text{grad } v\| d\lambda \\
 &\leq \left( \int_{B_r} \|\text{grad } u\|^2 d\lambda \right)^{1/2} \left( \int_{B_r} \|\text{grad } v\|^2 d\lambda \right)^{1/2} \\
 &= \left( \int_{S_r} u \frac{\partial u}{\partial n} d\tau - \int_{B_r} u \Delta u d\lambda \right)^{1/2} \left( \int_{S_r} v \frac{\partial v}{\partial n} d\tau - \int_{B_r} v \Delta v d\lambda \right)^{1/2}.
 \end{aligned}$$

Letting  $r \rightarrow 1$  and using (3.4) and (3.5) we obtain

$$(3.12) \quad [\mu, \nu] \leq \|\mu\| \|\nu\|.$$

Thus the proposition is true in this case.

Suppose that  $G\mu$  and  $G\nu$  are potentials whose measures are compactly supported, say in  $B_{1-\varepsilon}$  for some  $\varepsilon > 0$ . Thus  $G\mu$  and  $G\nu$  are harmonic on  $B - \bar{B}_{1-\varepsilon}$ . Let  $\{r_j\}$  be a sequence of numbers strictly decreasing to 0. Let  $\{\psi_j\}$  be  $C^\infty$  functions with support in  $B_{r_j} - \bar{B}_{r_{j+1}}$ , radial,  $\psi_j \geq 0$  and  $\int_B \psi_j d\lambda = 1$ . Let  $\delta > 0$  be so small that for  $|z| > 1 - \delta$ ,  $\phi_z(B_{r_j}) \subset B - \bar{B}_{1-\varepsilon}$ . Then precisely as in the proof of Proposition 2.1,  $\{G\mu * \psi_j\}$  and  $\{G\nu * \psi_j\}$  are harmonic on  $B - \bar{B}_{1-\delta}$ , they are  $C^\infty(B)$  and they increase respectively to  $G\mu$  and  $G\nu$  [U, Lemma 2.1]. Thus they are  $C^\infty$  potentials with compact support. From (3.12) we have

$$[G\mu * \psi_j, G\nu * \psi_j] \leq \|G\mu * \psi_j\| \|G\nu * \psi_j\|.$$

Letting  $j \rightarrow \infty$  and applying Lemma 3.1 gives

$$(3.13) \quad [\mu, \nu] \leq \|\mu\| \|\nu\|.$$

Finally, the proof of the proposition will be completed for arbitrary potentials  $G\mu$  and  $G\nu$  by considering the restrictions of  $\mu$  and  $\nu$  to  $B_{1-1/n}$  and applying (3.13) and Lemma 3.1.

**COROLLARY 3.3.** *Let  $\mu$  and  $\nu$  be two positive measures on  $B$ . Then  $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ .*

**PROOF.**

$$\begin{aligned}
 \|\mu + \nu\|^2 &= \|\mu\|^2 + \|\nu\|^2 + 2[\mu, \nu] \\
 &\leq \|\mu\|^2 + \|\nu\|^2 + 2\|\mu\| \|\nu\| = (\|\mu\| + \|\nu\|)^2.
 \end{aligned}$$



COROLLARY 3.4. Let  $\{G\mu_k\}$  be an increasing sequence of potentials such that  $\sup_k \|\mu_k\| < \infty$ . Then there is a potential  $G\mu$  such that  $\lim_{k \rightarrow \infty} G\mu_k = G\mu$  and  $\lim_{k \rightarrow \infty} \|\mu_k\| = \|\mu\|$ .

PROOF. Let  $0 < r < 1$ .  $B_r$  is regular for the Dirichlet problem on  $B_r$ . That is, for every continuous function  $f$  on  $S_r$ , there is a unique harmonic function  $P_r f$  on  $B_r$  which tends to  $f$  on  $S_r$  and which is  $\geq 0$  if  $f \geq 0$  [R, Lemma 5.5.4]. For  $\xi \in B_r$ , let  $\rho_\xi^r$  be the measure such that

$$\int_{S_r} f d\rho_\xi^r = P_r f(\xi)$$

[H, p. 426]. Then, by [U, Lemma 1.19],

$$\begin{aligned} (3.14) \quad \|\rho_\xi^r\|^2 &= \int_{S_r} G\rho_\xi^r d\rho_\xi^r \leq G\rho_\xi^r(\xi) \\ &= \int_{S_r} G(\xi, w) d\rho_\xi^r(w) = P_r G(\xi, \cdot)(\xi). \end{aligned}$$

Since  $P_r G(\xi, \cdot)$  decreases as  $r$  increases [U, Corollary 1.20],  $\lim_{r \rightarrow 1} P_r G(\xi, \cdot)$  defines a harmonic function on  $B$  [U, Proposition 1.10] which minorizes  $G(\xi, \cdot)$ . It follows the limit as  $r \rightarrow 1$  in (3.14) is 0. Thus  $\lim_{r \rightarrow 1} \|\rho_\xi^r\| = 0$ .

Let  $u = \lim_{k \rightarrow \infty} G\mu_k$ . Then

$$\int_{S_r} u d\rho_\xi^r = \lim_{k \rightarrow \infty} \int_{S_r} G\mu_k d\rho_\xi^r = \lim_{k \rightarrow \infty} [\mu_k, \rho_\xi^r] \leq \sup_{k > 0} \|\mu_k\| \|\rho_\xi^r\|,$$

by Proposition 3.2. Thus

$$(3.15) \quad \lim_{r \rightarrow 1} \int_{S_r} u d\rho_\xi^r = 0 \quad \text{for each } \xi \in B.$$

It is easy to see that the limit in (3.15) defines the greatest harmonic minorant of  $u$  on  $B$ . It follows that  $u$  is a potential. The result now follows from Lemma 3.1.

LEMMA 3.5. Let  $E$  be polar and let  $G\mu$  be a potential of finite energy. Then  $\mu(E) = 0$ .

PROOF. We may assume  $E$  is relatively compact. For each  $k > 0$  let  $A_k = \{z \in B | G\mu(z) \leq k\}$ . Let  $\mu_k$  be the restriction of  $\mu$  to  $A_k$ . Since  $A_k$  is closed,  $G\mu_k$  is harmonic on  $B \setminus A_k$ . For  $z \in A_k$ ,  $G\mu_k(z) \leq G\mu(z) \leq k$ . Thus  $G\mu_k$  is a finite potential and we may apply the Domination Principle to deduce  $G\mu_k \leq k$  on  $B$ .

Let  $p$  be a potential that is  $\infty$  on  $E$ . Let  $U$  be a relatively compact neighborhood of  $E$ . Then  $R_p^U$  is the potential of a measure  $\nu$  with support in  $\bar{U}$ . Since

$$\int_E G\nu d\mu_k \leq \int G\nu d\mu_k = \int G\mu_k d\nu \leq k \cdot \nu(\bar{U}) < \infty$$

and  $G\nu = \infty$  on  $E$ ,  $\mu_k(E)$  must be 0. Letting  $k \rightarrow \infty$  gives

$$\mu[E \cap \{z \in B | G\mu(z) < \infty\}] = 0.$$

If  $\mu[E \cap \{z \in B | G\mu = \infty\}]$  were positive, then

$$\infty = \int_E G\mu d\mu \leq \int G\mu d\mu = \|\mu\|^2,$$

contradicting our assumption. Thus  $\mu(E) = 0$ .

PROPOSITION 3.6. *Let  $\{G\mu_k\}$  be a decreasing sequence of potentials such that  $\|\mu_1\| < \infty$ . If  $G\mu = (\lim_{k \rightarrow \infty} G\mu_k)^\wedge$ , then  $\|\mu\| = \lim_{k \rightarrow \infty} \|\mu_k\|$ .*

PROOF. The Cartan-Brelot convergence theorem implies  $G\mu = \lim_{k \rightarrow \infty} G\mu_k$  except at most on a polar set. Using Lemma 3.5, the proof follows the same pattern as Lemma 3.1.

PROPOSITION 3.7. *Let  $E \subset B$  be polar. Then there is a potential of finite energy that is  $\infty$  on  $E$ .*

PROOF. Suppose first  $E$  is bounded. Since  $\hat{R}_1^E \equiv 0$ , there is a decreasing sequence of relatively compact open sets  $\{U_k\}$ , each containing  $E$  such that  $(\lim_{k \rightarrow \infty} R_1^{U_k})^\wedge \equiv 0$  (2.16b). Let  $\mu_k$  be the measure on  $\bar{U}_k$  whose potential is  $R_1^{U_k}$ . (Recall  $R_1^{U_k} = \hat{R}_1^{U_k}$  since  $U_k$  is open.) Then, by (2.14a),

$$\|\mu_k\|^2 = \int R_1^{U_k} d\mu_k \leq \int d\mu_k < \infty.$$

Thus  $\lim_{k \rightarrow \infty} \|\mu_k\| = 0$  (Proposition 3.6). Choose a subsequence  $\{\mu_{k_j}\}$  such that  $\|\mu_{k_j}\| < 1/2^j$ . Then Corollary 3.4 implies  $\sum_{j=1}^\infty R_1^{U_{k_j}}$  is a potential of finite energy. Clearly it is  $\infty$  on  $E$  (2.14b).

In general let  $\{E_k\}$  be an increasing sequence of bounded sets with union  $E$ . Choose a potential  $G\mu_k$  of finite energy that is  $\infty$  on  $E_k$ . Put  $\nu_k = \mu_k/2^k \|\mu_k\|$ . Then  $\sum G\nu_k$  is the required potential.

**4. Capacity.** Let  $E \subset B$ . Define

$$(4.1) \quad c(E) = \begin{cases} \infty & \text{if } \hat{R}_1^E \text{ is not a potential,} \\ \|\mu\|^2 & \text{if } \hat{R}_1^E = G\mu. \end{cases}$$

We call  $c(E)$  the *capacity* of  $E$ .

PROPOSITION 4.1. (a) *Let  $E \subset B$ . There is a decreasing sequence of open sets  $\{U_m\}$ , each containing  $E$  such that  $\lim_{m \rightarrow \infty} c(U_m) = c(E)$ .*

(b) *If  $E$  is Borel, there is an increasing sequence of compact sets  $\{K_m\}$ , each contained in  $E$  such that  $\lim_{m \rightarrow \infty} c(K_m) = c(E)$ .*

PROOF. (a) If  $c(E) = \infty$ , there is nothing to show. Suppose then  $c(E) < \infty$ . The result will follow from (2.16b) and Proposition 3.6 if we show there is an open set  $U$  containing  $E$  having finite capacity.

Let  $U_1 = \{z | \hat{R}_1^E(z) > \frac{1}{2}\}$ . Since  $R_1^E \equiv 1$  on  $E$ , (2.15) implies  $E \setminus U_1$  is polar. By Proposition 3.7, there is a potential  $G\nu$  of finite energy that is  $\infty$  on  $E \setminus U_1$ . Put  $U_2 = \{z | G\nu(z) > 1\}$ . Then  $U = U_1 \cup U_2$  is an open set containing  $E$ . Since

$$2\hat{R}_1^E(z) + G\nu(z) \geq 1 \quad (\text{all } z \in U),$$

we have

$$2\hat{R}_1^E(z) + G\nu(z) \geq R_1^U(z) \quad (\text{all } z \in B)$$

by definition of the reduced function. Since  $c(E) < \infty$ ,  $2\hat{R}_1^E$  has finite energy. Hence Corollary 3.3 implies so has  $R_1^U$ .

(b) The proof follows from (2.17b) and Lemma 3.1.

LEMMA 4.2. For each  $\xi \in B$  and  $0 < r < 1$ , the map

$$f(w) = \int_S G(\varphi_\xi(rz), w) d\sigma(z)$$

is continuous on  $B$ .

PROOF. It is clearly continuous at points of  $B \setminus \varphi_\xi(S_r)$ . If now  $w_0 \in \varphi_\xi(S_r)$ , then for each  $\varepsilon > 0$  the function

$$f_\varepsilon(w) = \int_S G(\varphi_\xi(r + \varepsilon)z, w) d\sigma(z)$$

is continuous at  $w_0$ . Since  $f_\varepsilon$  increases as  $\varepsilon$  decreases [U, Proposition 1.17, Corollary 1.20]  $f$  is lower semicontinuous at  $w_0$ . Repeating the argument with  $-\varepsilon$  completes the proof.

LEMMA 4.3. Let  $v$  be superharmonic on  $B$ . Then, for every  $\xi \in B$ ,

$$\lim_{r \rightarrow 0} \int_S v(\varphi_\xi(rz)) d\sigma(z) = v(\xi).$$

PROOF. The lower semicontinuity of  $v$  implies

$$\liminf_{r \rightarrow 0} \int_S v(\varphi_\xi(rz)) d\sigma(z) \geq v(\xi).$$

Super mean value property (2.4) implies

$$\limsup_{r \rightarrow 0} \int_S v(\varphi_\xi(rz)) d\sigma(z) \leq v(\xi).$$

PROPOSITION 4.4. Let  $E$  be a Borel, relatively compact subset of  $B$ . If  $\hat{R}_1^E$  is the potential  $G\mu$ , then  $\|\mu\|^2 = \mu(B)$ .

PROOF. Since  $\|\mu\|^2 = \int_B G\mu d\mu \leq \int_B 1 \cdot d\mu = \mu(B)$ , we may assume  $\|\mu\| < \infty$ . If  $E$  is closed, the result holds by Lemma 3.5 since, in general,  $\mu(B \setminus \bar{E}) = 0$ .

For a general  $E$ , choose  $\{K_j\}$  as in (2.17b). Put  $\hat{R}_1^{K_j} = G\mu_j$ . Then, by Lemma 3.1,

$$\lim_{j \rightarrow \infty} \mu_j(B) = \lim_{j \rightarrow \infty} \|\mu_j\|^2 = \|\mu\|^2.$$

By passing to a subsequence we may assume  $\{\mu_j\}$  converges weakly to a measure  $\nu$ . We show  $G\nu = \hat{R}_1^E$ .

$$\begin{aligned} \int_S \hat{R}_1^E(\varphi_\xi(rz)) d\sigma(z) &= \lim_{j \rightarrow \infty} \int_S G\mu_j(\varphi_\xi(rz)) d\sigma(z) \\ (4.2) \quad &= \lim_{j \rightarrow \infty} \int_S \int_{\bar{E}} G(\varphi_\xi(rz), w) d\mu_j(w) d\sigma(z) \\ &= \lim_{j \rightarrow \infty} \int_{\bar{E}} \int_S G(\varphi_\xi(rz), w) d\sigma(z) d\mu_j(w). \end{aligned}$$

Lemma 4.2 shows that the inner integral in (4.2) is a continuous function of  $w$ . Thus the limit is

$$\int_{\bar{E}} \int_S G(\varphi_\xi(rz), w) d\sigma(z) d\nu(w) = \int_S G\nu(\varphi_\xi(rz)) d\sigma(z).$$

Letting  $r \rightarrow 0$  in the first and last term in this string of equalities and applying Lemma 4.3 gives  $\hat{R}_1^E(\xi) = G\nu(\xi)$ .

Let  $\psi \in C_c^\infty(B)$  be identically 1 on a neighborhood of  $\bar{E}$ . Then, using Theorem 2.5 of [U],

$$\begin{aligned}\mu(B) &= \int_B \psi d\mu = - \int_B G\mu \Delta \psi d\lambda = - \int_B G\nu \Delta \psi d\lambda \\ &= \int_B \psi d\nu = \nu(B).\end{aligned}$$

Thus

$$\|\mu\|^2 = \lim_{j \rightarrow \infty} \|\mu_j\|^2 = \lim_{j \rightarrow \infty} \mu_j(B) = \nu(B) = \mu(B).$$

This concludes the proof.

## 5. Main results.

LEMMA 5.1. *Let  $\mu$  be a finite measure on  $B$  and let*

$$d\nu(w) = \frac{d\mu(w)}{(1 - |w|^2)^n}.$$

*For each  $z \in B$ , define  $S(z) = \varphi_z(B_{1/2})$ . Then*

$$\begin{aligned}(5.1) \quad &\lim_{z \rightarrow e} (1 - |z|^2)^n \int_{B \setminus S(z)} G(z, w) d\nu(w) \\ &= \lim_{z \rightarrow e} \int_{B \setminus S(z)} G(z, w) d\mu(w) = 0\end{aligned}$$

PROOF. If  $w \in B \setminus S(z)$ ,  $|\varphi_z(w)| \geq 1/2$ . Thus, by (2.12b),

$$\begin{aligned}(1 - |z|^2)^n G(z, w) &\leq c_2 (1 - |z|^2)^n (1 - |\varphi_z(w)|^2)^n \\ &= \frac{c_2 (1 - |z|^2)^{2n} (1 - |w|^2)^n}{|1 - \langle z, w \rangle|^{2n}} \\ &\leq \frac{c_2 (1 - |z|^2)^{2n} (1 - |w|^2)^n}{(1 - |z| |w|)^{2n}}.\end{aligned}$$

Thus

$$\begin{aligned}&\limsup_{z \rightarrow e} (1 - |z|^2)^n \int_{B \setminus S(z)} G(z, w) d\nu(w) \\ &\leq c_2 \limsup_{z \rightarrow e} (1 - |z|^2)^{2n} \int_B \frac{d\mu(w)}{(1 - |z| |w|)^{2n}} \\ &= 0\end{aligned}$$

by the bounded convergence theorem.

The last limit in (5.1) is 0, again by the bounded convergence theorem, since by (2.12b),

$$\chi_{B \setminus S(z)}(w) G(z, w) \leq c_2 \chi_{B \setminus S(z)}(w) (1 - |\varphi_z(w)|^2)^n \leq \left(\frac{3}{4}\right)^n c_2.$$

THEOREM 5.2. *Let  $E$  be a Borel subset of  $B$  with  $e = (1, 0, \dots, 0)$  as a limit point. The following are equivalent:*

(a) For every potential  $G\nu$ ,

$$\liminf_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n G\nu(z) = 0.$$

(b)  $\inf_{\varepsilon > 0} c(E \cap \{z \in B \mid |z - e| < \varepsilon\}) > 0$ .

REMARK 5.3. For each positive integer  $m$ , let  $U_m = \{z \in B \mid |z - e| < 1/m\}$ . Suppose for some  $m_0$ ,  $\hat{R}_1^{E \cap U_{m_0}}$  is a potential  $G\mu$ . For each  $m > m_0$  let  $\mu_m$  be the restriction of  $\mu$  to  $E \cap U_m$ . Since  $G\mu_m$  is harmonic on  $B \setminus \bar{U}_m$ ,  $\lim_{m \rightarrow \infty} G\mu_m$  is harmonic on  $B$  and minorizes  $G\mu$  [U, Proposition 1.10]. Thus  $\lim_{m \rightarrow \infty} G\mu_m$  is identically 0 on  $B$ . Proposition 3.6 therefore implies that either  $c(E \cap U_m) = \infty$  for all  $m$  or  $\lim_{m \rightarrow \infty} c(E \cap U_m) = 0$ . Thus (b) above is equivalent to  $c(E \cap U_m) = \infty$  for all  $m$ .

PROOF OF THE THEOREM. Suppose first that (b) fails. We will show there is a potential  $G\nu$  such that

$$(5.2) \quad \lim_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n G\nu(z) = \infty.$$

Remark 5.3 implies our assumption is equivalent to  $c(E \cap U_m) < \infty$  for some  $m$ . Proposition 4.1(a) implies there is an open set containing  $E \cap U_m$  and having finite capacity. Thus, for the purpose of finding  $\nu$  to satisfy (5.2), we may assume  $E$  is open.

Choose  $\{m_j\}$  increasing to  $\infty$  such that  $c(E \cap U_{m_j}) < 1/j^2 4^j$ . Put

$$V_j = E \cap \{z \in B \mid 1/m_{j+2} < |z - e| < 1/m_j\}.$$

Consider the potential  $G\mu_j = jR_1^{V_j}$ . Then  $\|\mu_j\|^2 = j^2 c(V_j) \leq 1/4^j$ . Thus Corollary 3.3 and Corollary 3.4 show  $\sum G\mu_j$  is a potential  $G\mu$  of finite energy. Since  $\mu = \sum \mu_j$ , Proposition 4.4 implies  $\mu(B) = \sum \mu_j(B) = \sum \|\mu_j\|^2 < \infty$ , so  $\mu$  is finite. Clearly

$$(5.3) \quad \lim_{\substack{z \rightarrow e \\ z \in E}} G\mu(z) = \infty.$$

Put  $d\nu(w) = d\mu(w)/(1 - |w|^2)^n$ . Let  $S(z) = \varphi_z(B_{1/2})$ . Lemma 5.1 and (5.3) imply

$$(5.4) \quad \lim_{\substack{z \rightarrow e \\ z \in E}} \int_{S(z)} G(z, w) d\mu(w) = \infty.$$

If  $w \in S(z)$ ,  $|\varphi_z(w)| \leq 1/2$ , hence

$$\begin{aligned} \frac{3}{4} &\leq 1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} \leq \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - |z| |w|)^2} \\ &\leq \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - |w|)^2} \leq \frac{4(1 - |z|^2)}{1 - |w|^2}. \end{aligned}$$

Thus

$$(5.5) \quad 1 - |z|^2 \geq \frac{3}{16}(1 - |w|^2) \quad (|\varphi_z(w)| \leq \frac{1}{2}).$$

Since  $|\varphi_z(w)| = |\varphi_w(z)|$ ,

$$(5.6) \quad 1 - |w|^2 \geq \frac{3}{16}(1 - |z|^2) \quad (|\varphi_z(w)| \leq \frac{1}{2}).$$

Thus

$$\begin{aligned}
 & \liminf_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n G\nu(z) \\
 & \geq \liminf_{\substack{z \rightarrow e \\ z \in E}} (1 - |z|^2)^n \int_{S(z)} G(z, w) d\nu(w) \\
 & \geq \left(\frac{3}{16}\right)^n \liminf_{\substack{z \rightarrow e \\ z \in E}} \int_{S(z)} G(z, w) d\mu(w) \\
 & = \infty
 \end{aligned}$$

by Lemma 5.1 and (5.3). This completes the proof of (a) $\Rightarrow$ (b).

Suppose now (b) holds. Consider a potential  $G\nu$ . Put  $d\mu(w) = (1 - |w|^2)^n d\nu(w)$ . Let

$$h(z) = \int_{S(z)} G(z, w) d\mu(w),$$

where  $S(z) = \varphi_z(B_{1/2})$ . According to Remark 5.3,  $c(E \cap U_j) = \infty$  for all  $j$ . Thus, by Proposition 4.1(b), there are compact sets  $E_j$  contained in  $U_j \cap E$  such that

$$(5.7) \quad \lim_{j \rightarrow \infty} c(E_j) = \infty.$$

Let  $G\mu_j = \hat{R}_1^{E_j}$  and let  $F_j = \bigcup_{z \in E_j} \varphi_z(B_{1/2})$ . Then, since  $|\varphi_z(w)| = |\varphi_w(z)|$ ,

$$\begin{aligned}
 (5.8) \quad \int h(z) d\mu_j(z) &= \int_{E_j} h(z) d\mu_j(z) \\
 &= \int_{E_j} \int_B \chi_{S(z)}(w) G(z, w) d\mu(w) d\mu_j(z) \\
 &= \int_{E_j} \int_B \chi_{S(w)}(z) G(z, w) d\mu(w) d\mu_j(z) \\
 &= \int_{F_j} \int_{E_j} G(z, w) d\mu_j(z) d\mu(w) \\
 &= \int_{F_j} \hat{R}_1^{E_j}(w) d\mu(w) \leq \mu(F_j)
 \end{aligned}$$

and the latter goes to 0 as  $j \rightarrow \infty$  since  $\chi_{F_j}(z) \rightarrow 0$  as  $|z| \rightarrow 1$  (2.1). This shows

$$\liminf_{\substack{z \rightarrow e \\ z \in E}} h(z) = 0,$$

for if  $h$  were bounded below by  $\varepsilon > 0$  on  $\bigcup_j E_j$ ,

$$\begin{aligned}
 \int h d\mu_j &= \int_{E_j} h d\mu_j \geq \varepsilon \mu_j(E_j) \\
 &= \varepsilon c(E_j) \quad (\text{Proposition 4.4})
 \end{aligned}$$

which tends to  $\infty$  as  $j \rightarrow \infty$  (5.7), contradicting the last inequality in (5.8). Thus

$$\begin{aligned} & \liminf_{\substack{z \rightarrow E \\ z \in E}} (1 - |z|^2)^n G\nu(z) \\ &= \liminf_{\substack{z \rightarrow E \\ z \in E}} (1 - |z|^2)^n \int_{S(z)} G(z, w) d\nu(w) \quad (\text{Lemma 5.1}) \\ &\leq \left(\frac{16}{3}\right)^n \liminf_{\substack{z \rightarrow E \\ z \in E}} \int_{S(z)} G(z, w) d\mu(w) \quad (5.6) \\ &= \left(\frac{16}{3}\right)^n \liminf_{\substack{z \rightarrow E \\ z \in E}} h(z) \\ &= 0. \end{aligned}$$

This completes the proof.

**6. Remarks.** Let  $0 < \delta \leq 1$ . Put

$$E_\delta = \{(z, z') \in \mathbb{C} \times \mathbb{C}^{n-1} \mid \operatorname{Im} z = 0, (\operatorname{Re} z)^2 + |z'|^2 < \delta\}.$$

In [HS] it is shown that the limit result of Theorem 5.2 holds with  $E = E_1$ . We show in this section directly that  $E_1$  satisfies condition (b) of Theorem 5.2.

**LEMMA 6.1.** *Let  $\varphi$  be an automorphism of  $B$  and  $E$  a Borel subset of  $B$ . Then  $c(\varphi(E)) = c(E)$ .*

**PROOF.** Suppose first that  $E$  is relatively compact. Since  $v \circ \varphi \in S^+$  whenever  $v \in S^+$  [U, Proposition 1.17], it follows from the definition that  $\hat{R}_1^{\varphi(E)}(z) = \hat{R}_1^E(\varphi(z))$ . Thus, if  $\hat{R}_1^E = G\mu$ ,

$$\begin{aligned} \hat{R}_1^{\varphi(E)}(z) &= \hat{R}_1^E(\varphi(z)) = \int_B G(\varphi(z), w) d\mu(w) \\ &= \int_B G(z, \varphi(w)) d\mu(w) \end{aligned}$$

since  $|\varphi_{\varphi(z)}(\varphi(w))| = |\varphi_z(w)| = G\nu(z)$ , where  $\nu = \mu \circ \varphi$ . Proposition 4.4 implies  $c(\varphi(E)) = \nu(\varphi(E)) = \mu(E) = c(E)$ .

In general, let  $E_r = E \cap B_r$ . Then  $c(\varphi(E)) = \lim_{r \rightarrow 1} c(\varphi(E_r)) = \lim_{r \rightarrow 1} c(E_r) = c(E)$ . The first and third equalities follow from Lemma 3.1 and the fact that if  $\{A_m\}$  is an increasing sequence of sets with union  $A$  then  $\lim_{m \rightarrow \infty} \hat{R}_1^{A_m} = \hat{R}_1^A$ .

**LEMMA 6.2.** *Let  $\hat{R}_1^E$  be a potential. Then  $E$  is polar if and only if  $c(E) = 0$ .*

**PROOF.** If  $E$  is polar,  $\hat{R}_1^E \equiv 0$ , hence  $c(E) = 0$ .

Suppose  $c(E) = 0$ . For every  $j$  there is an open set  $V_j$  containing  $E$  such that  $c(V_j) < 1/2^j$ . Then  $\sum \hat{R}_1^{V_j}$  is superharmonic (Corollary 3.3 and Corollary 3.4) and is  $\infty$  on  $E$ . This proves the Lemma.

We now show that  $E_1$  satisfies (b) of Theorem 5.2. For each  $\delta \in (0, 1)$ ,  $E_\delta$  is a  $(2n - 1)$ -dimensional disc, hence is not polar. (Indeed if it were polar, the fact that a finite union of polar sets is polar would allow us to construct an open region  $F$  whose boundary was polar. But a superharmonic function which was  $\infty$  on  $\partial F$  would be  $\infty$  on  $F$ , contradicting (iv) of the definition of superharmonic.) Thus  $c(E_\delta) > 0$ .

Let  $\{t_k\}$  be a sequence increasing to 1. Put  $z_k = (t_k, 0, \dots, 0)$ . Then  $\varphi_{z_k}(E_\delta) \subset E_1$ ,  $c(\varphi_{z_k}(E_\delta)) = c(E_\delta) > 0$  and since  $\varphi_{z_k}(E_\delta)$  moves out to  $e$  as  $k \rightarrow \infty$ , we see  $E_1$  satisfies (b) of Theorem 5.2.

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

DEPARTMENT OF MATHEMATICS, GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA 22030